Topology and its Applications 20 (1985) 251-255
North-Holland


# A THEOREM ON PLANAR CONTINUA AND AN APPLICATION TO AUTOMORPHISMS OF THE FIELD OF COMPLEX NUMBERS

Robert R. KALLMAN and Forest W. SIMMONS
*Department of Mathematics, North Texas State University, Denton, TX 76203, USA*

Received 15 December 1983

Let $K$ be a continuum in the plane which does not lie on a line. Then the set of differences, $K-K$, contains an open set. Let $\psi$ be an automorphism of the field of complex numbers which is bounded on an $F_\sigma$ set of positive inductive dimension. Then $\psi$ is continuous.

AMS (MOS) Subj. Class.: Primary 12D99, 54F20; Secondary 12D15, 12F20, 54F45

automorphism of the field of complex numbers
planar continua    inductive dimension

## 1. Introduction

It is well known from a first course in abstract algebra that any automorphism of the field of real numbers is the identity, but that the complex numbers have $2^c$ automorphisms, only two of which, namely the identity and complex conjugation, are continuous. In fact, the complex numbers have $2^c$ automorphisms which are bounded on a fixed compact perfect subset of the line. To see this, recall von Neumann's result [11] that there exists an uncountable analytic subset of the line consisting of numbers algebraically independent over the rationals. But a theorem of Souslin (see [6]) implies that any uncountable analytic set contains a compact perfect set. Hence, there exists a compact perfect subset $P$ of the line consisting of numbers algebraically independent over the rationals. Extend $P$ to a transcendence basis $B$ for the complexes $C$ over the rationals. $C$ is algebraic over $Q(B)$, and any field automorphism of $Q(B)$ therefore can be extended to be a field automorphism of $C$. One constructs automorphisms of $Q(B)$ merely by permuting the elements of $B$ around. By choosing automorphisms which permute the elements of $P$ among themselves, one obtains in this way $2^c$ distinct automorphisms of $C$ which leave $P$ invariant. Hence, one learns nothing from the fact that an automorphism of $C$ is bounded on a compact perfect set. But one may ask if some such condition does imply continuity.

The genesis of this paper is the following observation:

**Proposition 1.1.** *Let $\psi$ be an automorphism of $C$ which is bounded on a circle or on a line segment. Then $\psi$ is continuous.*

**Proof.** $\psi$ fixes the rationals. First assume that $\psi$ is bounded on some circle. By using one translation and one dilation, we may suppose that $\psi$ is bounded on $T$, the unit circle about 0. But then $\psi$ is bounded on $T+T=\{w+z \mid w$ and $z$ are in $T\}$, which, a simple geometric argument shows, is the closed disk of radius 2 about 0. Let $z$ lie in the unit disk about 0. The same is true for all the positive powers $z^n$ of $z$. Hence, there is some positive constant $D$ so that $|\psi(z^n)| \leq D$ for all $n \geq 1$. Take $n$th roots, let $n$ go to infinity, and conclude that $|\psi(z)| \leq 1$ for every $z$ in the unit disk about 0. Let $z_n$ be a sequence in $C$ so that $z_n \to 0$. Suppose that $|\psi(z_n)| \geq r > 0$, where $r$ is rational. Then $2z_n/r \to 0$ and $|\psi(2z_n/r)| \geq 2$. This is a contradiction. Hence, $\psi$ is continuous at 0 and so is continuous everywhere.

Next, if $\psi$ is bounded on some (closed) line segment, then by using a finite number of translations, dilations, and rotations (multiplication by a complex number of modulus one), we have that $\psi$ is bounded on the unit square $S$ with vertices $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. A simple geometric argument shows that $S+S-S=\{u+w-z \mid u, w$ and $z$ are in $S\}$ contains the interior of $S$. Hence, $\psi$ must be bounded on the interior of $S$. One concludes the proof of this case and Proposition 1.1 just as one did the previous case. □

Having proved this result, one might ask if it still holds if $\psi$ is bounded on a curve, or more generally, on a compact connected subset of the plane. The main purpose of this paper is to prove the following theorem:

**Theorem 1.2.** *Let $\psi$ be an automorphism of $C$ which is bounded on an $F_\sigma$ subset of the plane which has positive inductive dimension. Then $\psi$ is continuous.*

This result is proved in Section 3. In order to prove Theorem 1.2, a result on planar topology seems to be needed.

**Proposition 1.3.** *Let $K$ be a continuum in the plane which does not lie on a line. Then the set of differences, $K-K=\{w-z \mid w$ and $z$ are in $K\}$, contains an open set.*

This proposition is a fairly easy consequence of the chord theorem and the Baire category theorem, but does not seem to have been observed before. To an analyst, it is very reminiscent of Steinhaus' theorem, namely, that if $B$ is a measurable subset of a locally compact group with positive Haar measure, then $B^{-1} \cdot B$ contains a neighborhood of the identity. Proposition 1.3 is proved in Section 2.

Every nontrivial connected subset of the plane has positive dimension. One might ask if one can generalize Theorem 1.2 to the case in which $\psi$ is bounded on a nontrivial connected subset of the plane. This cannot be done. Dieudonne [3], modifying a construction of Livenson [8], showed that there is a connected subset $S$ of the plane consisting of numbers algebraically independent over $Q$. He even showed that the intersection of $S$ with every disk is connected and dense. As before, one can use those elements of $S$ which are in the unit disk about the origin to construct $2^c$ automorphisms of the complex numbers which are bounded on the intersection of $S$ with the unit disk about the origin. (See Wiesław's book [13, pp. 143–145] for a clear exposé of this construction.) In passing, one should note that one can combine Dieudonne's construction with the theory of real fields to construct a connected, dense, and locally connected subfield $F$ of $C$ so that $[C\colon F]=2$ and $F$ is not isomorphic to any subfield of the reals.

Finally, the real numbers contain no proper subfields of finite codimension, but an application of Galois theory shows that there exist subfields of the reals which are of countable codimension. A theorem is proved in Section 4 which shows that these subfields must be very bizarre.

## 2. Proof of Proposition 1.3

The proof of Proposition 1.3 is a fairly simple consequence of the chord theorem and the Baire category theorem. Two lemmas, based on the chord theorem, are proved first.

**Lemma 2.1.** *If $K$ is a planar continuum, then for each point $(p, q, r)$ of $K \times K \times [0, 1]$, there exists a point $(x, y, s)$ of $K \times K \times \{r, 1-r\}$ such that $y - x = s(q-p)$.*

**Proof.** Let $K$ and $(p, q, r)$ be as hypothesized. Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be a nested sequence of Peano continua in the plane whose intersection is $K$. For each natural number $n$, use the chord theorem [12, Theorem 15, p. 16] to pick a point $(x_n, y_n, s_n)$ in $K_n \times K_n \times \{r, 1-r\}$ such that $y_n - x_n = s_n(q-p)$. The compactness of $K_1 \times K_1 \times \{r, 1-r\}$ guarantees that the set $\{(x_n, y_n, s_n) \mid n \geq 1\}$ has a limit point. Any such point will serve as the desired triple $(x, y, s)$. This proves Lemma 2.1. □

**Lemma 2.2.** *If $p$ and $q$ are distinct points of the planar continuum $K$, then the set $K - K = \{x - y \mid x \text{ and } y \text{ are in } K\}$ contains a nondegenerate segment $J$ parallel to the line containing $p$ and $q$.*

**Proof.** Let $p$, $q$ and $K$ be as hypothesized. Let $A = \{s \text{ in } [0, 1] \mid s(q-p) = x - y$ for some $x$ and $y$ in $K\}$. $A$ is closed by a compactness argument. If $A$ contains a nondegenerate interval $I$, then the set $\{s(q-p) \mid s \text{ in } I\}$ will serve as the desired $J$. Hence, if $A$ contains the interval $[0, 1]$, there is nothing further to show. Otherwise, the set $[0, 1]$—$A$ is nonempty and open in $[0, 1]$, and therefore contains some nondegenerate interval $B$. In this case, the set $\{1 - r \mid r \text{ is in } B\}$ is an interval contained in $A$ by Lemma 2.1. This proves Lemma 2.2. □

The rest of the proof of Proposition 1.3 consists of an application of the Baire category theorem. First, find a point $p$ in $K$ for which $K-\{p\}$ is connected. Such a $p$ exists by a theorem of R.L. Moore (see Kuratowski [7, p. 177]). By translation if necessary, we may assume that $p$ is the origin. Let $A$ be the image of $K-\{(0,0)\}$ under the map $z \to z/|z|$, which takes the nonzero complex numbers continuously onto the standard unit circle. This connected set $A$ is a nontrivial subset of the circle since $K$ is not contained in a line. Let $A'$ be a nontrivial closed arc in $A$.

Next, choose a countable basis $V_n$ ($n \geq 1$) of open intervals for the usual topology of the real line. For each $n$, let $A_n = \{z \text{ in } A' \mid zV_n \subset K-K\}$. Each $A_n$ is closed, and their union is $A'$ by Lemma 2.2. The Baire category theorem implies that some $A_n$, say $A_m$, contains a nonempty open subarc $U$ of $A'$. The product $U \cdot V_m$ is the desired nonempty open subset of $K-K$. This proves Proposition 1.3. □

## 3. Proof of Theorem 1.2

The proof of Theorem 1.2 is a very easy corollary to hard facts in dimension theory and to Proposition 1.3.

Let $K$ be an $F_\sigma$ subset of the plane of positive dimension, and let $\psi$ be an automorphism of $C$ which is bounded on $K$. $K$ is a countable union of closed subsets. By the sum theorem [4, 1.3.1], one of these closed subsets must have positive dimension. But each closed set is a countable union of compact subsets, and so the sum theorem again implies that $\psi$ is bounded on some compact subset of the plane of positive dimension. But a theorem of Menger implies that a compact subset of the plane of positive dimension contains a nontrivial continuum (see [9, p. 209] or [7, Theorem 9, p. 172]). Hence, $\psi$ is bounded on a nontrivial continuum in the plane. We can assume that $K$ is this continuum. If $K$ lies on a line, then $K$ is a nontrivial line segment, and Theorem 1.2 follows from Proposition 1.1. If $K$ does not lie on a line, then $\psi$ is bounded on $K-K$, which contains an open set, and Theorem 1.2 again follows from Proposition 1.1. (Actually, to handle this second case, one did not need the full strength of Proposition 1.3, but only Lemma 2.2 plus Proposition 1.1.) This proves Theorem 1.2. □

Notice that the same techniques from dimension theory used in this proof enable one to extend Proposition 1.3 to the case in which $K$ is an $F_\sigma$ subset of the plane of positive inductive dimension.

## 4. Subfields on countable codimension

The only connection between this section and the other sections is that it deals with topological fields.

The real numbers contain no proper subfields of finite codimension [1, Satz 4]. But Białynicki-Birula [2] has shown, using Galois theory and the theory of topological groups, that there exist subfields of the reals which are of countable codimension. The following general theorem shows that these subfields must be quite bizarre:

**Theorem 4.1.** *Let $\Omega$ be a complete separable metric field, $F$ a subfield which is an analytic set, and suppose that $F$ is of at most countable codimension in $\Omega$. Then $F$ is closed, and some finite algebraic extension of $F$ is open in $\Omega$.*

**Proof.** Let $x_1 = 1, x_2, x_3, \ldots$ be a basis for $\Omega$ over $F$. Fix $m \geqslant 1$. Set $F_m = F + Fx_2 + Fx_3 + \cdots + Fx_m$. For each $n \geqslant m+1$, set $B_n = Fx_{m+1} + \cdots + Fx_n$. $F_m$ is an analytic subset of $\Omega$, for it is the continuous image of $F^m$. Similarly, each $B_n$ is an analytic subset of $\Omega$. Let $B = \bigcup_{n \geqslant m+1} B_n$, an analytic subset of $\Omega$. The mapping $(x, y) \to x + y$, $F_m \times B \to \Omega$, is continuous and one-to-one. This mapping is a Borel isomorphism by Souslin's theorem, and so $F_m$ and $B$ are Borel subsets of $\Omega$. $B$ is a transversal for the cosets of the quotient space $\Omega / F_m$, a quotient of additive abelian groups. Hence, $\Omega / F_m$ is countably separated. But then $F_m$ is closed [10, Theorem 1]. In particular, $F$ is closed. Since $\Omega = \bigcup_{m \geqslant 1} F_m$, the Baire category theorem implies that some $F_m$ is open. If $F$ is countable, then $\Omega$ is countable, and so $\Omega$ is discrete. But then $F$ is open. If $F$ is uncountable, then every element of $\Omega$ is algebraic over $F$. For if $x$ is transcendental over $F$, the elements $1/(x-a)$, $a$ in $F$, are uncountable and linearly independent over $F$. If $F_m$ is open in $\Omega$, then $F(x_2, \ldots, x_m)$ is a finite algebraic extension of $F$ which is open in $\Omega$. This proves Theorem 4.1. $\square$

## References


[1] E. Artin and O. Schreier, Eine Kennzeichnung der reell abgeschlossenen Koerper, Hamb. Abh. 5 (1927) 225-231.

[2] A. Białynicki-Birula, On subfields of countable codimension, Proc. Amer. Math. Soc. 35 (2) (1972) 354-356.

[3] J. Dieudonne, Sur les corps topologiques connexes, C. R. Acad. Paris Ser. A, B 221 (1945) 396-398.

[4] R. Engelking, Dimension Theory (North-Holland, Amsterdam, 1978).

[5] W. Hurewicz and H. Wallman, Dimension Theory (Princeton University Press, Princeton, NJ, 1941).

[6] K. Kuratowski, Topology, Vol. I (Academic Press, New York, 1966).

[7] K. Kuratowski, Topology, Vol. II (Academic Press, New York, 1968).

[8] E. Livenson, An example of a non-closed connected subgroup of the two-dimensional vector space, Ann. Math. 38 (1937) 920-922.

[9] K. Menger, Dimensionstheorie (Teubner, Leipzig, 1928).

[10] D.E. Miller, On the measurability of orbits in Borel actions, Proc. Amer. Math. Soc. 63 (1) (1977) 165-170.

[11] J. von Neumann, Ein System algebraisch unabhaengiger Zahlen, Math. Annalen 99 (1928) 134-141.

[12] D. Rolfsen, Knots and Links (Publish or Perish, Berkeley, CA, 1976).

[13] Wiesław, Topological Fields, Acta Universitatis Wratislaviensis, 675 (Wrocław, Poland, 1982).